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Divergence of the Bulk Resistance at Criticality in Disordered Media

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Consider the electrical resistance $r_n(p)$ of a hypercubic bond lattice $[0, n]^d$ in Z^d, where the bonds have resistance 1Ω with probability p or ∞ with probability $1-p$. Let $\rho_n(p) = n^{2-d} r_n(p)$ and $\rho(p) = \lim_{n \to \infty} \rho_n(p)$. It is well known that $\rho(p) < \infty$ if $p > p_c$ and $\rho(p) = \infty$ if $p < p_c$, where p_c is the percolation threshold. Here we show that $\rho(p_c) = \infty$, and $\lim_{p \downarrow p_c} \rho(p) = \rho(p_c) = \infty$.

KEY WORDS: Bulk resistance; criticality.

1. INTRODUCTION

The problem of bulk transport in disordered media has been studied by many researchers. It arises naturally in many fields such as biology and physics. Here we will focus on random resistor networks. The simple setup is as follows. We consider Z^d as a graph with edges connecting each pair of points with $\|x - y\| = 1$, where $\|x\| = \max_{1 \leqslant x \leqslant d} |x_i|$ for $x = (x_1, \ldots, x_d)$. Denote by E^d the edge set of Z^d. We allocate to each edge e of $E^d \cap [0, n]^d$, the electrical resistance $r(e)$. For simplicity, we assume that $\{r(e) : e \in E^d \cap [0, n]^d\}$ is an i.i.d. family with a common distribution as follows:

$$r(e) = \begin{cases} 1\Omega & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Let

$$L_n = \{0\} \times [0, n]^{d-1} \qquad \text{(the left hyperface of } [0, n]^d\text{)}$$

$$R_n = \{n\} \times [0, n]^{d-1} \qquad \text{(the right hyperface of } [0, n]^d\text{)}$$

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Now we connect all vertices in L_n and R_n by wires made from some super material which has zero electrical resistance. In other words, we can view all edges both in R_n and L_n as having the common endpoints R and L, respectively. Let r_n be the resistance between L_n and R_n after this identification of vertices in L_n and R_n. As sample space we take $\Omega = \prod_{r \in [0, n]^d \setminus (L_n \cup R_n)} \{1, \infty\}$ points of which are represented as $w = (r(e): e \in [0, n]^d \setminus (L_n \cap R_n))$ are called *configuration*. Let P_p be the corresponding product measure on Ω. Expectation with respect to P is denoted by E_p. We renormalize r_n by

$$\rho_n(p) = n^{d-2} r_n(p) \tag{1}$$

It is conjectured that ρ_n converges as $n \to \infty$. Since no one has proven the conjecture, we would rather assume that

$$\liminf \rho_n(p) = \rho(p)$$

We write $\rho(p)$ for the *bulk resistance*. The behavior of $\rho(p)$ is very similar to percolation. In fact, they even have the same critical threshold. To state it more precisely, consider standard (Bernoulli) bond percolation on Z^d, in which all edges are independently occupied corresponding to $r(e) = 1$ with probability p or vacant corresponding $r(e) = \infty$ with probability $1 - p$. The cluster of the vertex x, $C(x)$, consists of all vertices which are connected to x by an occupied path on Z^d. An occupied path here is a nearest neighbor path on Z^d whose edges are occupied. The percolation probability is defined to be

$$\theta(p) = P_p(|C(0)| = \infty)$$

and the critical threshold is defined to be

$$p_c = p_c(Z^d) = \sup\{p: \theta(p) = 0\}$$

It is well known that $0 < p_c < 1$. For any two sets of vertices A and B we write $A \leftrightarrow B$ for the event that there exists an occupied path from some vertex in A to some vertex in B. Denote by

$$H = \{(x_1, \dots, x_d): x_1 \geqslant 0\}$$

the half-space. Define the half-space percolation probability to be

$$\theta_H(p) = P_p((0, \dots, 0) \leftrightarrow \infty \text{ in } H)$$

With these definitions, Barsky *et al.*[1] proved the following remarkable result.

Proposition. $\theta_H(p)$ is continuous and $\theta_H(p_c) = 0$.

Now we return to discuss the bulk resistance. If p is below the critical threshold, by a standard percolation result (see Chapter 6 ref. 2), with probability less than $\exp(-Cn)$ there exists a path from L_n to R_n without using any bond e with $r(e)=\infty$, which is to say that

$$\rho(p)=\infty \qquad \text{if} \quad p<p_c$$

For $p>p_c$, Chayes and Chayes[3] proved that

$$\frac{1}{dp\theta^2(p)} \leqslant \rho(p) \leqslant c(p) \tag{2}$$

where $c(p)$ is a constant. In fact, one of most important problems in this field is to investigate the behavior when p is near or at p_c. In other words we at least want to know whether $\rho(p)$ blows up at p_c. Since it is widely believed that $\theta(p_c)=0$, it then follows from (2) that we can convince ourselves that $\rho(p_c)=\infty$. However, as far as we know, there is no rigorous proof to show that $\theta(p_c)=0$ except for $d=2$ and for a large d. To avoid the embarrassment here we improve the lower bound in (2) to $1/[d\theta_H(p)\,\theta(p)]$ as the following theorem.

Theorem. $1/[d\theta_H(p)\,\theta(p)] \leqslant \rho(p)$.

With the theorem and the proposition, it is easy to obtain the following corollary.

Corollary. $\rho(p_c)=\infty$ and $\lim_{p\downarrow p_c}\rho(p)=\infty$.

2. PROOF

Before the proof of the theorem, we first introduce some basic knowledge of electric networks.[4]

Shorting Law. Shorting certain sets of vertices on Z^d together can only decrease the effective resistance between two given vertices.

Proof of Theorem. The proof is similar to the proof in ref. 3. For simplicity we only prove the special case $d=3$. Extending the result to any $d \geqslant 2$ poses no serious difficulty. Set

$$A_i=\{(x_1,x_2,x_3)\in B(n)\colon x_1=i\}$$

i.e., the hyperplane with $x_1=i$ and

$$S_i=\{e\in E^d\cap[0,n]^d\colon v_1\in A_i,\, v_2\in A_{i+1},$$
$$\text{where } v_1 \text{ and } v_2 \text{ are the two vertices of } e\}$$

For each configuration w and each $0 \leqslant i \leqslant n$, we consider the following set:

$$T_i(w, n) = \{e \in S_i\text{: there exist two disjoint paths with } r(e) = 1 \text{ from two vertices of } e \text{ to } L_n \text{ and } R_n\text{, respectively, such that one of these two paths from the vertex } e \text{ in } A_{i+1} \text{ to } R_n \text{ stays in } x_1 > i\}$$

and

$$V_i^L(w, n) = \{v\text{: the vertices of } T_i(w, n) \text{ in } A_i\}$$

$$V_i^R(w, n) = \{v\text{: the vertices of } T_i(w, n) \text{ in } A_{i+1}\}$$

(see Fig. 1). Now we connect all vertices in $V_i^L(w, n)$ for $i = 0, 1, 2,..., n-1$ by conducting wires (see Fig. 1). Then we connect all vertices in $V_i^R(w, n)$ for $i = 1,..., n$ by conducting wires. Furthermore, if $V_{i+1}^L \cap V_i^R = \emptyset$, we also connect one of V_{i+1}^L to one of V_i^R by a conducting wire (see Fig. 1). Denote by r_n' the resistance with the special connection. By the shorting law,

$$r_n \geqslant r_n'$$

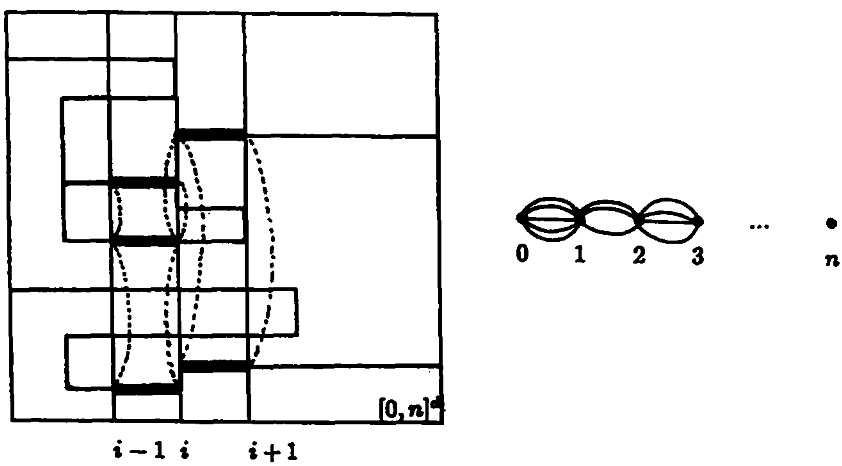

Fig. 1. The bold edges on the left side are $T_{i-1}(w, n)$ and $T_i(w, n)$, respectively, and the dotted lines are conducting wires. Right: The graph after identifying the vertices of V_{i+1}^L and V_i^R as a vertex.

Now we estimate r'_n. Let $r'_n(i)$ be the resistance between V_i^L and V_i^R. Surely, for each i there are many paths with $r(e)=1$ from V_i^R that first cross the plane A_i and then double back to join V_{i+1}^L (see Fig. 1). Note that by our special connections, V_i^R and V_{i+1}^L are connected by conducting wires so that these paths will not reduce any resistance. Therefore, we may identify these vertices of V_{i+1}^L and these vertices of V_i^R as a vertex. If we do this, we obtain a graph which has vertices denoted by $0, 1,..., n$ and edges between i and $i+1$ denoted by $e_1,..., e_{|V_i^L|}$ (see Fig. 1). Clearly, there are $|V_i^L|$ edges between vertex i and vertex $i+1$ with resistance 1. Then

$$r'_n = \sum_{i=0}^{n-1} r'_n(i) \qquad \text{and} \qquad r'_n(i) = \frac{1}{|V_i^L|} \tag{3}$$

Therefore, by Jensen's inequality,

$$r'_n \geqslant \frac{n^2}{\sum_{i=0}^{n-1} |V_i^L|} \tag{4}$$

By the definition of V_i^L and the same proof as in ref. 3 [see (4.8)–(4.11) in ref. 3],

$$\begin{aligned} \frac{1}{\rho(p)} &\leqslant \limsup \frac{1}{nr_n} \leqslant \limsup \frac{1}{nr'_n} \\ &\leqslant \limsup \frac{\sum_{i=0}^{n-1} |V_i^L|}{n^3} \leqslant d\theta_H(p)\, \theta(p) \end{aligned} \tag{5}$$

The theorem is proved.

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